AIAA JOURNAL
Vol. 33, No. 8, August 1995

# Improved Integrated Structural Control Design Using Covariance Control Parameterization

J. B. Layton*
*Clarkson University, Potsdam, New York 13699*
and
L. D. Peterson†
*University of Colorado, Boulder, Colorado 80309-0429*

**This paper investigates multiobjective integrated control/structure design using covariance control to parameterize the controller. An advantage of this approach is that the control objectives can be specified as a set of competing analytical goals or constraints instead of a single scalar cost function. From this formulation, a multiobjective numerical optimization problem is formulated to execute the integrated control-structure design. The performance of this approach is investigated on a simple disturbance rejection problem in which the control objectives and the structural design objectives conflict. The integrated design optimization both increases control performance and decreases the structural mass with a counterintuitive design. In addition, these results add to previous work that demonstrates the potential of integrated design optimization of structures and controllers. It then appears that when the structure and controller are considered simultaneously, the resulting combination is an improved design. However, the extent of the design improvement is apparently problem dependent.**

## Nomenclature

| | |
|---|---|
| $F_k$ | = objective functions |
| $f_{\max}$ | = maximum value of the $K_f$ functions |
| $g_k$ | = constraints of the form $g_j(p) \leq 0$ |
| $I$ | = appropriately sized identity matrix |
| $I_{n_c}$ | = $n_c \times n_c$ identity matrix |
| $N_{\text{con}}$ | = number of constraints |
| $N_{\text{obj}}$ | = number of objective functions |
| $N_S$ | = number of structural design variables |
| $p$ | = vector of design variables |
| $u$ | = vector of actuator forces (length $n_u$) |
| $w$ | = vector of disturbances (length $n_w$) |
| $x_c$ | = controller state vector |
| $x_p$ | = system state vector (length $n_x$) |
| $y$ | = vector of controlled outputs (length $n_y$) |
| $z$ | = vector representing output of the sensors (length $n_z$) |
| $\sigma(\,)$ | = singular value function |

## Introduction

THE idea that it is possible to numerically resolve conflicting design requirements between structures and control systems has been previously investigated.[1–4] For both space structures and aeroservoelastic systems, the structural objective to minimize mass and/or maximize stiffness often decreases the ultimate performance achieved by the control system. For instance, decreasing the structural inertia can make disturbance rejection by the controller more difficult, whereas increasing the stiffness can also increase the control effort required to maintain a quasistatic shape. Integrated design optimization seeks to quantitatively resolve such design conflicts by simultaneously considering the structural and control design objectives within a unified numerical optimization procedure. Clearly, considering all these different objectives within a numerical procedure is a complex, perhaps impractical, task.

A traditional method for accommodating this multiobjective problem is to decouple the control and the structural designs and allow each discipline to proceed without communication to the other. This often results in excessively conservative designs, because each design group must make conservative assumptions about the results of the other's design. Multiobjective design techniques seek to codify the design processes of distinct disciplines under a single mathematical construct. The trend represented by these methods is generally to expand the problem away from a single, composite objective function into physically meaningful individual objectives. Our research seeks to advance a step further by expanding the control part of the problem to be more than a single scalar cost function itself and to couple a structural objective into the problem.

Our formulation of the combined structure/control problem is based on covariance control theory, which has been developed by Yasuda and Skelton[5] and Skelton et al.[6] In covariance control, all reduced-order linear stabilizing controllers are parameterized by the matrix elements of the closed-loop covariance matrix $X$. In a previous paper, we developed a nonlinear multiobjective control formulation using this description of the controller.[7] We adapted covariance control theory for numerical optimization by further parameterizing $X$ by its Cholesky decomposition. This allows the reduced-order control problem and its associated multiple design objectives to be expressed as analytical functions of the design parameters. In this paper, we couple the covariance parameterization of the controller to a structural design problem to investigate the efficacy of this technique for interdisciplinary, multiobjective design searches.

## Theoretical Development

Consider a linear time-invariant system with the following state-space equations:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + D_p w \\ z &= M_p x_p \\ y &= C_p x_p \end{aligned} \tag{1}$$

Any continuous, linear feedback controller of order $n_c$ can be defined by

$$\begin{aligned} \dot{x}_c &= A_c x_c + F z \\ u &= K x_c + H z \end{aligned} \tag{2}$$

Received March 28, 1994; revision received Nov. 7, 1994; accepted for publication Nov. 8, 1994. 

*Assistant Professor, Department of Mechanical and Aeronautical Engineering. Member AIAA.

†Assistant Professor, Center for Space Structures and Control and Department of Aerospace Engineering Sciences. Senior Member AIAA.

The matrices in this model can be grouped into a single feedback gain matrix $G$:

$$G = \begin{bmatrix} H & K \\ F & A_c \end{bmatrix} \tag{3}$$

The covariance of the closed-loop system is defined to be the one-at-a-time (OAT) covariance, which is defined as follows:

$$X_{OAT} = \sum_{i=1}^{n_x+n_c} \int_0^\infty x(i,t)x^T(i,t)\,dt \tag{4}$$

where $x(i, t)$ is defined to be the state response of the system to the $i$th excitation of the following set:

$$x(0) = \left\{ \begin{matrix} 0 \\ \vdots \\ x_j(0) \\ \vdots \\ 0 \end{matrix} \right\} j = 1, \ldots, n_x$$
$$w(t) = \left\{ \begin{matrix} 0 \\ \vdots \\ w_k\delta(t) \\ \vdots \\ 0 \end{matrix} \right\} k = 1, \ldots, n_w \tag{5}$$

The closed-loop covariance $X$ can then be computed using the following Lyapunov equation:

$$(A + BGM)X + X(A + BGM)^T + DWD^T + X_0 = 0 \tag{6}$$

where $A$, $B$, $M$, and $D$ are defined as

$$A = \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix} \quad D = \begin{bmatrix} D_p \\ 0 \end{bmatrix}$$
$$M = \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix} \tag{7}$$

The closed-loop state vector $x$ implicit in Eq. (6) includes both the plant states and the controller states

$$x = \begin{bmatrix} x_p \\ x_c \end{bmatrix} \tag{8}$$

so that the partitions of $X$ are

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix} \tag{9}$$

The basic idea of covariance control theory is that $G$ can be thought of as a function of $X$. Reference 5 shows that it is possible to solve Eq. (6) for $G$ as a function of $X$. This solution explicitly parameterizes all reduced-order linear stabilizing controllers that assign the covariance matrix $X$ to the closed-loop system. Using this so-called $X$ parameterization, all reduced-order stabilizing controllers can be parameterized by the following equation relating $X$ to $G$:

$$G = -\tfrac{1}{2}B^+Q(2I - BB^+)X^{-1}M^+ + \tfrac{1}{2}B^+(\psi^T - \psi)BB^+X^{-1}M^+$$
$$+ B^+XM^+M(I - \Gamma\Gamma^+)S(I - \Gamma\Gamma^+)M^+ = G_1 + G_2SG_3 \tag{10}$$

in which $Q$, $\psi$, $L$, and $\Gamma$ are defined as

$$Q = XA^T + AX + DWD^T + X_0$$
$$\psi = L^+(I - M^+M)X^{-1}Q(2I - M^+M) + QL^+M$$
$$L = (I - M^+M)X^{-1}BB^+ \tag{11}$$
$$\Gamma = M^+M(I - BB^+)$$

where + denotes the Moore–Penrose inverse. This particular $G$ is said to assign $X$ to the closed-loop system. The skew-symmetric matrix $S$ in Eq. (10) can be uniquely defined as a function of $X$ to minimize control effort, and so it is considered to be a dependent design variable that is a function of the remaining design variables constituting $X$.[6]

For this $G$ to exist, the following two equality constraints must be satisfied by $X$ and its partitions[5]:

$$C_1(X_p) = (I - B_pB_p^+)Q_p(I - B_pB_p^+) = 0$$
$$C_2(X) = (I - M_p^+M_p)\bar{Q}_p(I - M_p^+M_p) = 0 \tag{12}$$

in which $Q_p$ and $\bar{Q}_p$ are defined as

$$Q_p = X_pA_p^T + A_pX_p + D_pWD_p^T + X_{0_p}$$
$$\bar{Q}_p = \bar{X}_p^{-1}\bar{F}\bar{X}_p^{-1}$$
$$\bar{X}_p = X_p - X_{pc}X_c^{-1}X_{pc}^T$$

$$\bar{F} = \bar{X}_pA_p^T + A_p\bar{X}_p + D_pWD_p^T + X_{0_p}$$
$$+ X_{pc}X_c^{-1}X_{0_c}X_c^{-1}X_{pc}^T \tag{13}$$

$$X_0 = \begin{bmatrix} X_{0_p} & 0 \\ 0 & X_{0_c} \end{bmatrix}$$

where $X_{0_p}$ and $X_{0_c}$ are appropriately dimensioned initial covariance matrices. The two equality constraints $C_1(X_p)$ and $C_2(X)$ are referred to as the covariance assignability conditions in covariance control theory. They serve the role of the controllability and observability constraints that are implicit in the linear quadratic Gaussian (LQG) formulation. As noted in Ref. 7, these equality constraints are quickly satisfied in the numerical optimization procedure, because they define an apparently large null space of the covariance design parameters.

Within the context of covariance control theory as defined earlier, the control design problem can be stated as follows:

Find a positive definite $X$ subject to system performance constraints and covariance assignability conditions.

There are several equivalent tests for a matrix to determine positive definiteness. The most common test stipulates that all of the eigenvalues must be positive. For $X > 0$,

$$\lambda(X) > 0 \tag{14}$$

Since it is desirable to use a gradient-based optimization scheme, the gradients of the eigenvalues would be needed.

Finding these gradients can be numerically difficult. Haug and Rousselet have shown that the eigenvalues are (Frechet) differentiable if they are distinct.[8] On the other hand, if some of the eigenvalues are repeated, then they are only directionally (Gateaux) differentiable. Many of the methods available for determining the gradient of repeated eigenvalues over a subspace either use simplifying assumptions to create a tractable problem or are so general that they are prohibitive in terms of CPU time.[4]

The Cholesky decomposition can be used to further parameterize $X$ so that the positive definite constraint is simplified. This approach uses the elements of the lower triangular matrix from the Cholesky decomposition of $X$,

$$X = LL^T \tag{15}$$

where $L$ is the matrix, in which the lower triangular portion is nonzero, with the dimension of $L$ being $N = n_x + n_c$. From Ref. 3, if the diagonal elements of $L$ are positive, then the matrix $X$ will be positive definite. If the design variables are changed so that the diagonal elements of $L$ are always positive, then no constraint on $L$ will be needed to guarantee positive definiteness of $X$. This eliminates

Table 1 Comparison of final designs for the two-bar truss

| | Control effort reduction, % | Output cost reduction 1, % | Output cost reduction 2, % | Total structural mass | Mass 1 | Mass 2 | Number of iterations |
|---|---|---|---|---|---|---|---|
| Control-only design | 58 | 21.75 | 8.66 | 1500 | 500 | 1000 | 1200 |
| Integrated design | 86 | 14.8 | 9.2 | 1492 (0.53%) | 443 (11.25%) | 1049 (−4.95%) | 1000 |

the positive definite constraint on $X$. To guarantee that the diagonal elements are always positive, the diagonal design variables are squared so that

$$L = \begin{bmatrix} q_{11}^2 & 0 & \cdots & 0 \\ q_{21} & q_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \cdots & q_{NN}^2 \end{bmatrix} \tag{16}$$

The control design problem is then stated as follows:

Find $q_{ij}$, $i, j = 1, \ldots, N$, subject to system performance constraints and covariance assignability conditions.

The preceding parameterization of the controller is combined with structural design variables ($A_i$), objectives, and constraints to form a multidisciplinary, multiobjective nonlinear programming problem as follows:

Minimize control effort $V$ and mass $M$ where

$$V = \mathrm{tr}\big(HM_pX_pM_p^TH^TR + HM_pX_{pc}K^TR + KX_{pc}^TM_p^TH^TR + KX_cK^TR\big)$$

$$M = M(A_i)$$

over the design space

$$q_{ij}, \qquad i, j = 1, \ldots, N = (n_x + n_c) \qquad A_i, i = 1, \ldots, N_S$$

subject to

$$\sigma\big(C_pX_pC_p^T\big) < \bar{\sigma} \qquad i = 1, \ldots, n_y$$

$$C_1(X_p) = 0 \tag{17}$$

$$C_2(X) = 0$$

The output cost defined in Eq. (17) is the singular values of the system output covariance $Y = C_pX_pC_p^T$. This choice of output cost is consistent with that defined in Ref. 9. It has the effect that it limits the peak output of the system to be below some upper limit $\bar{\sigma}$.

In this study, we solved the optimization problem using the multiobjective constrained optimizer KSOPT.[10] It uses the Kreisselmeier–Steinhauser ($KS$) function to create a single surface combining all of the constraints and objectives into a single variable geometry surface. The $KS$ function is defined by

$$KS(\boldsymbol{p}) = f_{\max} + \frac{1}{\rho}\ell n \sum_{k=1}^{K_f} \exp[\rho f_k(\boldsymbol{p}) - f_{\max}] \tag{18}$$

where

$$K_f = N_{\text{obj}} + N_{\text{con}} \tag{19}$$

and

$$f_k = \begin{cases} \boldsymbol{F}_k & k = 1, \ldots, N_{\text{obj}} \\ \boldsymbol{g}_k & k = N_{\text{obj}} + 1, \ldots, N_{\text{obj}} + N_{\text{con}} \end{cases} \tag{20}$$

The KSOPT algorithm belongs in the class of sequential unconstrained minimization techniques because $\rho$ is increased as the optimal point is approached to emphasize the largest contribution to the $KS$ cost function. This asymptotically transforms the objective function(s) into goal constraint(s). As KSOPT searches for the minimum value of the $KS$ function, the design point is moved towards a compromise minimum. In representing the objective function(s) as goal constraint(s), the constrained compromise minimum satisfies the classical definition of a locally pareto-optimal point. Reference 10 shows the relationship between this method and goal programming, proving its ability to find a pareto-optimal point.

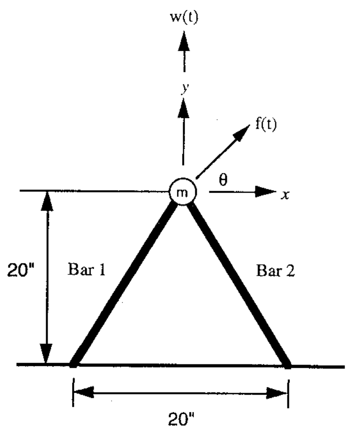


Fig. 1 Two-bar truss configuration.

One of the advantages of the covariance parameterization is that the gradients required for the numerical optimization procedure can be analytically computed. This allows the gradients of the objective function $V$ and the constraints to be computed as required by the optimization procedure. Because of the complexity of the resulting equations, however, these expressions are not presented here. Instead, the reader is referred to Ref. 4.

## Numerical Results

The previously discussed concepts were used to design a control/structure system for a simple flexible system. This problem was selected so that it demonstrates the feasibility of the formulation. More importantly, the example was selected so that the results could be easily and critically evaluated through insights into the physical behavior of the system.

This problem examined here is a common example in the control/structure optimization literature. It is a two-bar truss with an additional mass at the end of the truss. The configuration of the truss is shown in Fig. 1. The equations of motion are given in Ref. 2. The bar cross-sectional areas and Young's modulus are scaled in this example, but the natural frequencies of the scaled model are consistent with an actual structure of this size and concentrated mass. The scaling forces the material mass density to become so small ($2.59 \times 10^{-6}$ lb-s$^2$/in.$^4$) that the mass of the bars is negligible compared with the concentrated mass for the dynamic equations of motion. However, the cross-sectional areas of the bars are a direct measure of the actual truss structure weight. Therefore the sum of the cross-sectional areas becomes the structural objective-function with the structural design variables being the cross-sectional areas of the two truss bars.

For this fourth-order system we will attempt to find a second-order controller. The velocity of the mass in both the $x$ and $y$ directions is used for feedback, with a disturbance in the $y$ direction and a sensor and an actuator for both the $x$ and $y$ directions (collocated sensor/actuator). The outputs of the system are the $x$ and $y$ displacements of the concentrated mass, which result in two additional control constraints (termed output cost constraints). The

**Table 2 Comparison of final designs for the two-bar truss with those of Ref. 2**

| | Control effort reduction (from initial value), % | Output cost reduction $x$, % | Output cost reduction $y$, % | Total structural mass reduction, % |
|---|---|---|---|---|
| Ref. 2 | 3.3 | 4.5 | 4 | −2 |
| Integrated design | 86 | 14.8 | 9.2 | 0.53 |

initial design is chosen to be a rate feedback controller derived from a full-order LQG controller with the following arbitrarily chosen weighting matrices:

State weighting:

$$Q = I$$

Control effort weighting:

$$R = 1.0x10^{-2}I \tag{21}$$

The system performance output goals were to be reduced by 10% from their initial closed-loop value, but the KSOPT procedure was allowed to continue to search for a pareto-optimal point. Overall there are 23 design variables and 22 constraints for the problem.

Table 1 presents the results for the optimization problem. It includes the results for the problem when the structural variables are not included (noted as control-only design) and the problem when structural design variables are included (noted as integrated design), which are both generated from the same starting point. In examining the results it can be seen that the optimizer took an interesting approach to satisfying the conflicting requirements for the integrated case. Although the total structural mass was changed very little, the algorithm redistributed the mass to better achieve all of the objectives. By decreasing the area of the first bar and increasing the area of the second bar, it was able to reduce the control effort by 86% from its starting value, to decrease the $x$ displacement output cost ($x$ singular value of the output covariance) by 14.8%, and to decrease the $y$ displacement output cost ($y$ singular value of the output covariance) by 9.2%. It appears that the optimizer redistributed the mass to satisfy the competing objectives and constraints and to find a pareto-optimal point. Comparing with the control-only design, one can see that including the structural degrees of freedom allows the optimizer to further reduce the control effort while exceeding the output cost constraint requirements.

The small changes in the cross-sectional areas agree with previous results, which considered the same problem but with an LQG controller, recognizing that the optimization problem is different.[2] However, the general optimization problems are similar, and the results of Ref. 2 are intended to support the results of this paper. No direct comparisons between the results of this paper and those of Ref. 2 will be made. In that reference several different control/structure problems were considered. The one that is most similar to our approach is design case 3, which uses normalized objectives and constraints. Table 2 compares our results with those in Ref. 2. The results presented here and those of Ref. 2 have similar levels of performance improvement. Consequently, confidence is gained in the formulation presented here.

It is important to interpret these numerical optimization results in the context of more physically meaningful control parameters. This was done by comparing the multi-input/multi-output bode plots of the closed-loop system for the initial and final designs. From the phase information in these plots, it is obvious that the design only slightly lowered the frequency of the second (vertical bounce) mode so that the first (lateral vibration) mode is gain stabilized. This is done without affecting the static structural stiffness. Then, the controller introduces a substantial (150-deg) phase lead in the frequencies between the two modal resonances, thus improving the broadband performance of the close-loop system. It is interesting to note that a sequential structural and control design would not have attempted to lower the frequency of the second structural mode, and so it would never have even allowed the possibility of a significant phase lead at intermediate frequencies. Only an integrated design recognizes this possibility, leading to a more rational design.

## Conclusions

This paper demonstrated the benefits of integrated control/structure design using covariance control to parameterize the controller. An advantage of covariance control is that the control objectives can be specified as a set of competing analytical goals or constraints instead of a single scalar cost function. In addition, the controller order is specified a priori and need not match the order of the system model. Using the Cholesky parameterization of the covariance controller, the structural mass was modified and the controller performance was improved. These results demonstrate the sensitivity of integrated design methods to proper parameterization of the controller. In addition, these results add to previous work that demonstrates the potential of integrated design optimization of structures and controllers. It then appears that when the structure and controller are considered simultaneously, the resulting combination is an improved design. However, the extent of the design improvements are apparently problem dependent.

## References


[1]Khot, N. S., Eastep, F. E., and Venkayya, V. B., "Simultaneous Optimal Structure/Control Modifications to Enhance the Vibration Control of a Large Flexible Structure," AIAA Paper 85-1925, Aug. 1985.

[2]Gilbert, M. G., and Schmidt, D. K., "Integrated Structure/Control Law Design by Multilevel Decomposition," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 1001–1007.

[3]Junkins, J. L., and Rew, D. W., "Unified Optimization of Structures and Controllers," *Large Space Structures: Dynamics and Control*, Springer–Verlag, New York, 1988, pp. 323–353.

[4]Layton, J. B., "Multiobjective Control/Structure Design Optimization in the Presence of Practical Constraints," Ph.D. Dissertation, School of Aeronautical and Astronautical Engineering, Purdue Univ., Lafayette, IN, May 1992.

[5]Yasuda, K., and Skelton, R. E., "Assigning Controllability and Observability Grammians in Feedback Control," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 878–885.

[6]Skelton, R. E., Xu, J. H., and Yasuda, K., "On the-Freedom in Covariance Control," AIAA Paper 90-3471, Aug. 1990.

[7]Peterson, L. D., and Layton, J. B., "Covariance Control Parameterization for Combined Optimization of Structures and Controllers," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, 1994, pp. 1364–1366.

[8]Haug, E. J., and Rouselet, B., "Design Sensitivity Analysis in Structural Mechanics. II. Eigenvalue Variations," *Journal of Structural Mechanics*, Vol. 8, No. 2, 1980, pp. 161–186.

[9]Peterson, L. D., "Bounding the Transient Response of Structures to Uncertain Disturbances," AIAA Paper 91-1021, April 1991.

[10]Sobieszczanski-Sobieski, J., Dovi, A. R., and Wrenn, G. A., "A New Algorithm for the General Multiobjective Optimization," AIAA Paper 88-2434, April 1988.